§ Fundamental Theorems for Curves (do Carmo § 1.5)

Question: In general, does the curvature $k : I \to \mathbb{R}$ determine the curve $\alpha : I \to \mathbb{R}^2$ (p.b.a.l.) "completely" (up to rigid motions) ? YES!

Fundamental Theorem of Plane Curves

Given a smooth function $k : I \to \mathbb{R}$,
$\exists \; \alpha : I \longrightarrow \mathbb{R}^2$ p.b.a.l. (defined on the same I)
s.t. $k_\alpha(s) = k(s) \quad \forall s \in I$

Moreover, α is unique up to orientation-preserving rigid motions.

Note: The basic idea is that $k_\alpha \approx \alpha''$ (but non-linear!)

$$k_\alpha \approx \alpha'' \xrightarrow{\text{integrate}} \alpha' = T \xrightarrow{\text{integrate}} \alpha$$

ambiguity by $\varphi = A\vec{x} + b$ "integration constants"

Proof: (I) Existence

Fix $s_0 \in I$, define (Recall: $\theta' = k$)

$$\theta(s) := \int_{s_0}^{s} k(u)\,du$$

hence if we set

$$\alpha'(s) = T(s) = \underbrace{(\cos\theta(s), \sin\theta(s))}_{\text{unit vector}}$$

Integrating gives

$$\alpha(s) = \left(\int_{s_0}^{s} \cos\theta(t)\,dt \;,\; \int_{s_0}^{s} \sin\theta(t)\,dt \right)$$

Exercise: Check α is p.b.a.l. and $k_\alpha(s) = k(s)$.

(II) Uniqueness

Suppose $\beta : I \to \mathbb{R}^2$ is another curve p.b.a.l. s.t.

$$k_\beta(s) = k(s) = k_\alpha(s). \quad \forall s \in I.$$

Fix $s_0 \in I$. Consider the Frenet frames

$$\{T_\alpha(s), N_\alpha(s)\} \quad \text{and} \quad \{T_\beta(s), N_\beta(s)\}$$

$\exists$ unique orientation-preserving rigid motion

$$\varphi(x) = Ax + b$$

s.t. (1) $\varphi(\alpha(s_0)) = \beta(s_0)$ "match the point"

(2) $\begin{cases} A(T_\alpha(s_0)) = T_\alpha(s_0) \\ A(N_\beta(s_0)) = N_\beta(s_0) \end{cases}$ "match the frame"

<u>Claim</u>: $\varphi \circ \alpha = \beta$

Consider $f(s) = |T_{\varphi\circ\alpha}(s) - T_\beta(s)|^2$, $s \in I$.

$$= |A\, T_\alpha(s) - T_\beta(s)|^2$$

Differentiate in s, applying Frenet equations,

$$\frac{1}{2} f' = \langle A T_\alpha' - T_\beta' , A T_\alpha - T_\beta \rangle$$

$$= \langle A(k_\alpha N_\alpha) - k_\beta N_\beta , A T_\alpha - T_\beta \rangle$$

$(k_\alpha = k = k_\beta)$ $\quad = k \langle A N_\alpha - N_\beta , A T_\alpha - T_\beta \rangle$

$$= k\big(\langle A N_\alpha , A T_\alpha \rangle + \langle N_\beta , T_\beta \rangle$$

$$- \langle A N_\alpha , T_\beta \rangle - \langle N_\beta , A T_\alpha \rangle \big)$$

<u>Note</u>: • $\langle N_\beta, T_\beta \rangle = 0$ since $N_\beta \perp T_\beta$

• $\langle A N_\alpha, A T_\alpha \rangle \underset{\substack{\uparrow \\ \because A \in SO(2)}}{=} \langle N_\alpha, T_\alpha \rangle = 0$

• $$\begin{aligned} \langle A N_\alpha, T_\beta \rangle &= \langle A J T_\alpha, T_\beta \rangle \\ &= \langle J A T_\alpha, T_\beta \rangle \quad (\because AJ = JA) \text{ why?} \\ &= \langle J^2 A T_\alpha, J T_\beta \rangle \quad (\because J \in SO(2)) \\ &= -\langle A T_\alpha, N_\beta \rangle \quad (\because J^2 = -1) \end{aligned}$$

Combining these calculations, we have

$$f'(s) \equiv 0 \qquad \forall s \in I$$

Since $f(s_0) = 0$ by the choice of φ, $f(s) \equiv 0 \;\; \forall s \in I$.

Therefore,

$$(\varphi \circ \alpha)'(s) = T_{\varphi \circ \alpha}(s) = T_\beta(s) = \beta'(s) \qquad \forall s \in I$$

Integrating s and using $\varphi(\alpha(s_0)) = \beta(s_0)$,

$$\Rightarrow \qquad (\varphi \circ \alpha)(s) = \beta(s) \qquad \forall s \in I.$$

This completes the proof!

$\square$

Application: The only plane curves with $k \equiv$ constant are straight lines and circles.

$(k \equiv 0)$ $\quad\quad$ $(k \equiv \pm \frac{1}{r})$

depends on orientation

Q: What about space curves?

Fundamental Theorem of Space Curves

Given smooth functions $k, \tau : I \to \mathbb{R}$ with $k > 0$, there exists a space curve $\alpha : I \to \mathbb{R}^3$ p.b.a.l. s.t.

$$k_\alpha = k \quad \text{and} \quad \tau_\alpha = \tau$$

Moreover, α is unique up to orientation-preserving rigid motions of $\mathbb{R}^3$.

Proof: Fix $s_0 \in I$, and any frame $\{T_0, N_0, B_0\}$ of $\mathbb{R}^3$.

Consider the Frenet equations

$$(\#): \begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \underbrace{\begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}}_{\text{given}} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

1st order system linear ODEs

By fundamental existence theorem of ODEs,

$\exists$ solution $T(s), N(s), B(s)$, $s \in I$ to (#)

with "initial condition": $\{T(s_0), N(s_0), B(s_0)\} = \{T_0, N_0, B_0\}$

Claim: $\{T(s), N(s), B(s)\}$ is a frame $\forall s \in I$.

Proof: Define a 3×3 matrix

$$M(s) = \begin{pmatrix} - & T(s) & - \\ - & N(s) & - \\ - & B(s) & - \end{pmatrix}$$

It suffices to check:

$$M(s) \in SO(3) \quad \forall s \in I.$$

Known: $M(s_0) \in SO(3)$ since $\{T_0, N_0, B_0\}$ is a frame.

Define $Q = M M^T$ (depends on $s \in I$)

Check: Q satisfies the following ODE:

$$(*) \begin{cases} Q' = KQ - QK \\ Q(s_0) = I \end{cases}$$

We can rewrite the Frenet equations (#) in

matrix form: $\boxed{M' = KM}$

where $K = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix}$ is "skew-symmetric" $\boxed{K^T = -K}$

Therefore,

$$\begin{aligned} Q' &= M'M^T + M(M')^T \\ &= KMM^T + MM^TK^T \\ &= KQ - QK \end{aligned}$$

Note that $Q(s) \equiv I$ is a solution to $(*)$.
By Fundamental uniqueness of ODEs, it must be the only solution. So $MM^T \equiv I \quad \forall s \in I$.
By continuity, $\det M \equiv 1 \quad \forall s \in I$, so $M(s) \in SO(3)$ for all $s \in I$. This proves the claim.

Now, once we know

$$\{T(s), N(s), B(s)\} \text{ is a frame } \forall s \in I.$$

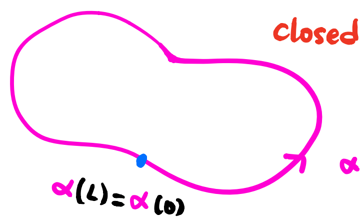
We can integrate $\alpha' = T$ to obtain. The rest of the details and the uniqueness part are left as an exercise.

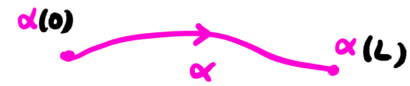
$\square$

§ Some global theorems for plane curves (do Carmo §1.7)

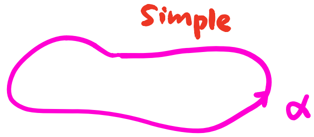
<u>Def$\underline{^n}$</u>: A plane curve $\alpha : [0, L] \to \mathbb{R}^2$ is said to be

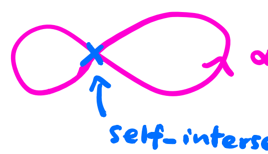
- closed if $\alpha^{(k)}(0) = \alpha^{(k)}(L)$ for $k = 0, 1, 2, \ldots$



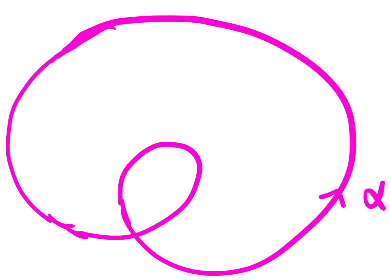


- Simple if α is 1-1 except possibly at $s = L$.



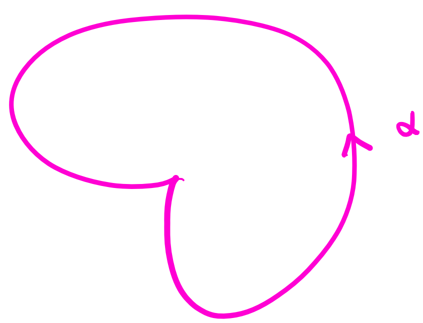


<u>Examples</u>



closed

~~simple~~



~~closed~~

simple

Theorem of Turning Tangents

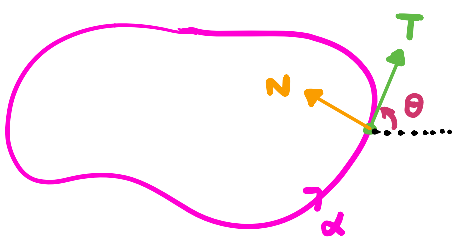
For any simple closed curve α in $\mathbb{R}^2$, which is positively oriented (ie. N points inward)

$$\boxed{\int_\alpha k(s)\, ds = 2\pi}$$

Proof: Let $\theta(s)$ be the angle from the positive x-axis to the unit tangent $T(s) = \alpha'(s)$, where α is p.b.a.l.

$$\boxed{\theta'(s) = k(s)}$$

$$\Downarrow$$



$$\underbrace{\theta(L) - \theta(0)}_{= 2\pi} = \int_0^L k(s)\, ds$$

$\square$

Remarks: (1) If α is negatively oriented, then

$$\int_\alpha k(s)\, ds = -2\pi$$

(2) In general,

if α is not simple, then

$$\int_\alpha k(s)\, ds = 2\pi \underset{\in \mathbb{Z}}{m}$$

rotation index / winding number